Quantum thermodynamics of systems with anomalous dissipative coupling

Alessandro Cuccoli,* Andrea Fubini,† and Valerio Tognetti‡
Dipartimento di Fisica dell'Università di Firenze and Istituto Nazionale di Fisica della Materia (INFM), Via G. Sansone 1, I-50019 Sesto Fiorentino (FI), Italy

Ruggero Vaia§
Istituto di Elettronica Quantistica del Consiglio Nazionale delle Ricerche, via Panciatichi 56/30, I-50127 Firenze, Italy and Istituto Nazionale di Fisica della Materia (INFM)

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The standard *system-plus-reservoir* approach used in the study of dissipative systems can be meaningfully generalized to a dissipative coupling involving the momentum, instead of the coordinate: the corresponding equation of motion differs from the Langevin equation, so this is called *anomalous* dissipation. It occurs for systems where such coupling can indeed be derived from the physical analysis of the degrees of freedom that can be treated as a dissipation bath. Starting from the influence functional corresponding to anomalous dissipation, it is shown how to derive the effective classical potential that gives the quantum thermal averages for the dissipative system in terms of classical-like calculations; the generalization to many degrees of freedom is given. The formalism is applied to a single particle in a double well and to the discrete ϕ^4 model. At variance with the standard case, the fluctuations of the coordinate are enhanced by anomalous dissipative coupling.

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I. INTRODUCTION

The *system-plus-reservoir* (SPR) model [1–5] is the most common and effective approach to the treatment of quantum dissipation. This model realistically assumes the dissipation in a system as due to the interaction with a reservoir (or bath, or environment): anyone of the numerous degrees of freedom of the reservoir is only weakly perturbed, so the reservoir is at thermal equilibrium; moreover, in order to reproduce the general phenomenological dynamics (quantum Langevin equation for the coordinate) it is sufficient to assume a linear interaction and a bath of harmonic oscillators [4], so that in the imaginary-time path-integral formalism it is possible to integrate out the bath variables and get a reduced description of the system in terms of a bilocal *influence action* [5]. Starting from this framework, in previous papers we obtained the classical effective potential for the calculation of thermal averages in a dissipative quantum system [6] and gave applications to the ϕ^4-chain model [7] and to a two-dimensional array of Josephson junctions [8].

At variance with the standard approach of coupling the bath with the system's coordinate, we consider in this paper the complementary possibility of a coupling with the momentum, that Leggett called *anomalous dissipation* [9]. In this case the dynamics can be reduced to a pseudo-Langevin equation where the dissipative term contains the second derivative of the potential. This raises the question whether one can recognize the dissipation mechanism just from the phenomenological behavior: we believe that one should have a physically meaningful microscopic model for the bath rather than rely upon the phenomenological counterpart. The case of anomalous dissipation occurs, for instance, when the effect of the blackbody electromagnetic field on a Josephson junction is considered [10].

As for thermodynamics, we will show that the *anomalous influence action* can be derived following the original Feynman-Vernon [11] idea, which, due to the momentum-path dependence, can be pursued at the price of involving the full phase-space path integral.

The main goal of this paper is in the treatment of this path integral within the effective-potential formalism [12] that reduces the evaluation of quantum-dissipative thermal averages to much simpler classical-like configuration integrals. This is accomplished also in the case of many degrees of freedom, making quantitative calculations feasible. In general, the inclusion of dissipation by coupling the system's momenta with the environment results in higher quantum fluctuations of the coordinates, while those of the momenta are suppressed.

We introduce in Sec. II the concept of anomalous dissipation; it turns out that the influence action depends on the momentum path and it is hence necessary to use the Hamiltonian path integral. In Sec. III we treat it within the pure-quantum self-consistent harmonic approximation [12], deriving the classical effective potential and the corresponding classical-like expressions for quantum thermal averages; in the case of many degrees of freedom the treatment is simplified by the additional ''low-coupling'' approximation. Eventually, in Sec. IV we show how the framework works for the single particle in a double-well potential and for the ϕ^4 chain.

II. FROM STANDARD TO ANOMALOUS DISSIPATION

The *standard* SPR Hamiltonian has the form

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + V(\hat{q}) + \frac{1}{2}\sum_{\ell}\left[\frac{\hat{p}_\ell^2}{m_\ell} + m_\ell\omega_\ell^2(\hat{q}_\ell - \hat{q})^2\right], \quad (1)$$

*Electronic address: cuccoli@fi.infn.it
†Electronic address: fubini@fi.infn.it
‡Electronic address: tognetti@fi.infn.it
§Electronic address: vaia@fi.cnr.it

where $\hat{p}$ and $\hat{q}$ are the momentum and coordinate of the ''system'' (here, one single particle in the one-dimensional potential V), while $\hat{p}_\ell$ and $\hat{q}_\ell$ are those of the ℓth degree of freedom of the reservoir. It is quite a general result [4] that a suitable distribution of the bath parameters $\{m_\ell, \omega_\ell\}$ (see Appendix A) can reproduce, when the bath variables are eliminated from the equations of motion, the most general quantum Langevin equation, namely,

$$m\ddot{\hat{q}} + \int_{-\infty}^{t} dt'\, \gamma(t-t')\dot{\hat{q}}(t') + V'(\hat{q}) = \hat{F}(t). \tag{2}$$

Since it does not explicitly contain the ''microscopic'' degrees of freedom, the Langevin equation constitutes a macroscopic description of the dissipative system with a clear phenomenological meaning, in the sense that the memory function $\gamma(t)$ can be thought to be determined experimentally. The thermodynamic density matrix for the standard SPR model at the equilibrium temperature $T=\beta^{-1}$ has the path-integral expression (A2), where dissipation is described by the additional *influence action*

$$S_{\mathrm{I}}[q(u)] = \int_0^{\beta\hbar} \frac{du}{2\hbar} \int_0^{\beta\hbar} \frac{du'}{\beta\hbar} k(u-u') q(u) q(u'), \tag{3}$$

and the Matsubara transform of the kernel $k(u)$ is directly related to the Laplace transform of the memory function as $k_n = |\nu_n| \tilde{\gamma}(|\nu_n|)$, where $\nu_n = 2\pi n/\beta\hbar$. For a harmonic potential one obtains for the density matrix the exact result (A5), so it appears that standard dissipation quenches $\langle \hat{q}^2 \rangle$ and rises $\langle \hat{p}^2 \rangle$. This breaks the canonical symmetry between the coordinate and the momentum, and is obviously a consequence of the bath being coupled to the coordinate in the standard SPR model (1). In the general case of a nonlinear interaction $V(q)$ the evaluation of the path integral (A2) for the SPR model becomes quite complicated and was the subject of Refs. [6] and [7], where a classical effective potential suitable to reduce the problem to classical-like expressions was introduced as a generalization of the original approach of Refs. [13] and [14].

The different case of *anomalous* dissipation occurs when the SPR model is modified including the momentum in the place of the coordinate in the interaction with the reservoir. The dependence upon the bath coordinates and momenta is quadratic, so they can be exchanged at will and the *anomalous* SPR Hamiltonian can be written

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + V(\hat{q}) + \frac{1}{2} \sum_\ell \left[\frac{(\hat{p}_\ell - \hat{p})^2}{m_\ell} + m_\ell \omega_\ell^2 \hat{q}_\ell^2 \right]. \tag{4}$$

It is immediately seen that Eq. (22) of Ref. [10], regarding a single Josephson junction interacting with the blackbody electromagnetic field (in the dipole approximation), can be cast exactly in this form, constituting a first physical example of the relevance of treating anomalous dissipation.

Let us look at the dynamics of the system (4): as in the standard SPR case, one can derive the equations of motion and then eliminate the bath variables. The result is a pseudo-Langevin equation,

$$m\ddot{\hat{q}} + m \int_{-\infty}^{t} dt'\, \eta(t-t') \partial_{t'} V'(\hat{q}(t')) + V'(\hat{q}) = \hat{F}(t). \tag{5}$$

To interpret Eq. (5), observe that in the classical limit one has $\partial_{t'} V'(q(t')) = V''(q(t'))\dot{q}(t')$, so that in the neighborhood of the stable minimum, where $V''(q) \sim m\omega^2$ tends to a positive constant, it reduces to the standard form of the Langevin equation (2); for a harmonic potential this always occurs. The physical difference is that the full damping function $\gamma(t) = m^2\omega^2 \eta(t)$ depends on the system's mass and potential, not barely on the reservoir's characteristics, so that the frictional force depends on how the system's motion arises, which prevents a simple phenomenological interpretation [4]. On the same footing, the concept of anomalous dissipation was considered by Caldeira and Leggett (see Appendix C of Ref. [3]) and further analyzed by Leggett [9]. Their purpose was to establish that in the context of tunneling problems the standard SPR model (1) (with a possible coupling to a nonlinear function of the coordinate) is the most general one to be considered in order to infer the effects of damping from a knowledge of the phenomenological quasiclassical dissipative equation (2); therefore, they rule out the anomalous case as it can give a negative friction coefficient ($V'' < 0$) in the pseudo-Langevin equation over much of the tunneling region, which would lead to qualitatively different results. On the other hand, since the pseudo-Langevin equation reduces to the Langevin one in the dynamical asymptotic region (i.e., close to the equilibrium configuration), a purely phenomenological approach to dissipation based on the dynamical linear response cannot distinguish whether the underlying dissipative mechanism is of the standard or of the anomalous type, while this distinction plays a fundamental role in the quantum statistical mechanics. However, rather than being a schematization for a phenomenological dissipative behavior, the mechanism can happen to be determined from a physically sound microscopic model that can have the form (4): once such microscopic SPR Hamiltonian is known, there is no point in trying to reduce it to a phenomenological description. Although examples can be given of how anomalous dissipation can arise just reinterpreting the canonical variables in a standard SPR system (e.g., a linear dissipative system as a simple RLC circuit [9], using the canonical symmetry between magnetic flux and charge), it is more significant that the anomalous dissipative mechanism is obtained on a *microscopic* basis for Josephson junctions, when quantum electromagnetic fluctuations are considered in the dipole approximation [10].

In order to get a first insight about the dissipation effect onto the thermodynamics, let us consider the harmonic oscillator. In Appendix B the analytic solution is easily obtained using the canonical symmetry between coordinate and momentum. At variance with the standard case, it appears that anomalous dissipation quenches the fluctuation of the mo-

mentum, Eq. (B9), and enhances those of the coordinate, Eq. (B10), as it is expected since the environment now performs a ''measurement'' of the momentum. This kind of behavior is expected also when the system is subjected to a nonlinear potential [9], but in this case the evaluation of quantum-dissipative thermal averages is by no means straightforward, and one has to resort to some approximation scheme. The purpose of the following Section is indeed to derive the effective classical potential and the related prescriptions in order to evaluate the system's thermodynamic averages for the anomalous SPR model (4).

In writing the path integral for the density matrix corresponding to the Hamiltonian (4), it appears that in order to integrate out the bath variables one must resort to the full phase-space formulation for the system, so that the momentum path $p(u)$ enters the interaction with the bath. Indeed, the *anomalous* influence action turns out to be (see Appendix C)

$$S_{\mathrm{I}}[p(u)]=\int_0^{\beta\hbar}\frac{du}{2\hbar}\int_0^{\beta\hbar}\frac{du'}{\beta\hbar}\kappa(u-u')p(u)p(u'), \quad (6)$$

and the kernel $\kappa(u)=\Sigma_n\kappa_n e^{i\nu_n u}$ is now directly related to the Laplace transform of the memory function $\eta(t)$ as

$$\kappa_n=|\nu_n|\tilde{\eta}(|\nu_n|). \quad (7)$$

Note that $\kappa(u)$ is periodic and, since $\kappa_n=\kappa_{-n}$, also even, one has the properties $\kappa(u)=\kappa(-u)=\kappa(u+\beta\hbar)$; in addition $\kappa_0=0$, meaning that the influence action does not include a local component, which trivially renormalizes the mass.

For the phase-space path integral it is convenient to use the symmetric (Weyl) ordering prescription [15,16]. The Weyl symbol for the density operator can, therefore, be written as

$$\rho(p,q)=\int\mathcal{D}[p_u,q_u]\exp\{-S[p_u,q_u]-S_{\mathrm{I}}[p_u]\}, \quad (8)$$

where for compactness the arguments are indicated as subscripts, and the nondissipative part of the system's action reads

$$\begin{aligned}S[p_u,q_u]=&\int_0^{\beta\hbar}\frac{du}{\hbar}\left[\frac{i}{2}(q_u\dot{p}_u-p_u\dot{q}_u)+\frac{p_u^2}{2m}+V(q_u)\right]\\&+\frac{i}{2\hbar}(q_0p_\beta-p_0q_\beta)+\frac{i}{\hbar}[(q_\beta-q_0)p\\&-(p_\beta-p_0)q]. \quad (9)\end{aligned}$$

Note that the arguments p and q do appear explicitly in the action, while there are no constraints upon the paths; in particular, the end points $(p_0,\ p_\beta,\ q_0,\ q_\beta)$ are also integrated over [15]. The thermal average of an observable $\hat{\mathcal{O}}$ can be written in terms of its Weyl symbol

$$\mathcal{O}(p,q)=\int dx e^{-ipx/\hbar}\left\langle q+\frac{x}{2}\middle|\hat{\mathcal{O}}\middle|q-\frac{x}{2}\right\rangle \quad (10)$$

as

$$\langle\hat{\mathcal{O}}\rangle=\frac{1}{\mathcal{Z}}\int\frac{dpdq}{2\pi\hbar}\rho(p,q)\mathcal{O}(p,q). \quad (11)$$

III. EFFECTIVE POTENTIAL

A. One degree of freedom

The strategy [12] to approximate the density (8) is to use the quadratic trial action $S_0[p_u,q_u;\overline{q}]$ expressed as Eq. (9), with $V(q_u)$ replaced by the trial ''potential''

$$V_0(q_u;\overline{q})=w(\overline{q})+\frac{1}{2}m\omega^2(\overline{q})(q_u-\overline{q})^2, \quad (12)$$

which is indeed a functional through its dependence on the path's average point

$$\overline{q}[q_u]=\int_0^{\beta\hbar}\frac{du}{\beta\hbar}q_u, \quad (13)$$

and to optimize its parameters $w(\overline{q})$ and $\omega^2(\overline{q})$.

The reduced density

$$\begin{aligned}\overline{\rho}_0(p,q;\overline{q})=&\int\mathcal{D}[p_u,q_u]\delta\left(\overline{q}-\int_0^{\beta\hbar}\frac{du}{\beta\hbar}q_u\right)\\&\times\exp\{-S_0[p_u,q_u;\overline{q}]-S_{\mathrm{I}}[p_u]\} \quad (14)\end{aligned}$$

collects the contribution of classes of paths that share the same average point $\overline{q}$, so the path integral (8) for the trial action S_0 can be cast into the form

$$\rho_0(p,q)=\int d\overline{q}\,\overline{\rho}_0(p,q;\overline{q}). \quad (15)$$

What we are going to do is to optimize the reduced density $\overline{\rho}$. This is an important point, that justifies the accuracy of the approximation: we choose the best approximating Gaussian distribution for the *separate* contribution of each class of paths (labeled by $\overline{q}$), which describes the purely quantum-dissipative contribution to the fluctuations around any ''classical'' configuration $\overline{q}$. In other words, the classical fluctuations are accounted for exactly.

The reduced density can be evaluated analytically: by including the constraint for the variable $\overline{q}$ in the action through the Fourier representation of the Dirac δ function, it is possible to get an integral expression in terms of the dissipative harmonic oscillator density matrix. In Appendix D we report the actual calculations that lead to the final result, which is a Gaussian distribution in (p,q),

$$\overline{\rho}_0(p,q;\overline{q})=\sqrt{\frac{2\pi m}{\beta}}\,e^{-\beta V_{\mathrm{eff}}(\overline{q})}\frac{e^{-p^2/2\lambda}}{\sqrt{2\pi\lambda}}\,\frac{e^{-(q-\overline{q})^2/2\alpha}}{\sqrt{2\pi\alpha}}, \quad (16)$$

where the effective potential reads

$$V_{\rm eff}(\overline{q}) = w(\overline{q}) + \frac{1}{\beta}\mu(\overline{q}), \tag{17}$$

with

$$\mu(\overline{q}) = \sum_{n=1}^{\infty} \ln \frac{\nu_n^2 + (1+m\kappa_n)\omega^2(\overline{q})}{\nu_n^2}, \tag{18}$$

and the relevant variances are

$$\lambda(\overline{q}) = \frac{m}{\beta} \sum_{n=-\infty}^{\infty} \frac{\omega^2(\overline{q})}{\nu_n^2 + (1+m\kappa_n)\omega^2(\overline{q})}, \tag{19}$$

$$\alpha(\overline{q}) = \frac{2}{m\beta} \sum_{n=1}^{\infty} \frac{1+m\kappa_n}{\nu_n^2 + (1+m\kappa_n)\omega^2(\overline{q})}; \tag{20}$$

apart from the dependence on the variational parameter $\omega^2(\overline{q})$, these variances have the harmonic-oscillator forms (B9) and (B10) with the important difference that α lacks the $n=0$ term, i.e., its classical contribution.

We introduce, henceforth, the coordinate fluctuation variable $\xi = q - \overline{q}$ in the place of q and the following double-bracket notation for the Gaussian averages over the variables p and ξ defined by $\overline{\rho_0}$:

$$\langle\langle p^2 \rangle\rangle = \lambda(\overline{q}) \langle\langle \xi^2 \rangle\rangle = \alpha(\overline{q}). \tag{21}$$

The physical interpretation of the above formulas becomes transparent if one expresses the resulting approximation for the thermal average (11) using Eqs. (15) and (16), namely,

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{\mathcal{Z}} \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int d\overline{q} e^{-\beta V_{\rm eff}(\overline{q})} \langle\langle O(p, \overline{q}+\xi) \rangle\rangle; \tag{22}$$

the average over ξ accounts for the nonclassical fluctuations of the coordinate, while the classical part is accounted for exactly by the classical-like expression for the thermodynamic average; the average over p takes into account the full fluctuation, since its classical contribution is exactly Gaussian. The detailed discussion made in Ref. [12] applies to this case as well, provided one takes into account that the contribution of the dissipative nonlocal action is also accounted for by the variances λ and α.

The last point concerns the optimization of the parameters $w(\overline{q})$ and $\omega^2(\overline{q})$. This can be performed imposing that the trial and the true potential (and their derivatives up to the second one) have the same $\overline{\rho_0}$ average [16,12]

$$\langle\langle V(\overline{q}+\xi) \rangle\rangle = w(\overline{q}) + \frac{1}{2} m\omega^2(\overline{q})\alpha(\overline{q}), \tag{23}$$

$$\langle\langle V''(\overline{q}+\xi) \rangle\rangle = m\omega^2(\overline{q}), \tag{24}$$

while the condition for the first derivative is trivial, since a linear term $\sim(q-\overline{q})$ in V_0 does not contribute to the action by the very definition of the average point. Equation (24) is a self-consistent equation that identifies the parameter $\omega^2(\overline{q})$, while the determination of $w(\overline{q})$ in Eq. (23) permits to rewrite the effective potential as

$$V_{\rm eff}(\overline{q}) = \langle\langle V(\overline{q}+\xi) \rangle\rangle - \frac{1}{2} m\omega^2(\overline{q})\alpha(\overline{q}) + \frac{1}{\beta}\mu(\overline{q}). \tag{25}$$

The differential operator

$$\Delta = \frac{1}{2}\alpha(\overline{q})\partial_{\overline{q}}^2, \tag{26}$$

understood as not operating onto its coefficient, is such that $\Delta F(\overline{q}) = \langle\langle F(\overline{q}+\xi) \rangle\rangle$, and allows us to rewrite the effective potential in a more compact way,

$$V_{\rm eff}(\overline{q}) = (1-\Delta) e^{\Delta} V(\overline{q}) + \frac{1}{\beta}\mu(\overline{q}). \tag{27}$$

It appears that the first term gives corrections of order α^2 to the potential, while first order renormalizations arise from the last term.

B. Many degrees of freedom

Let us now consider a general system with N degrees of freedom, i.e., canonical coordinate and momentum operators $\hat{\boldsymbol{q}} = \{\hat{q}_i\}_{i=1,\ldots,N}$ and $\hat{\boldsymbol{p}} = \{\hat{p}_i\}_{i=1,\ldots,N}$, with the commutation relations $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$ (we set $\hbar = 1$ in this section), and described by a Hamiltonian with a quadratic kinetic energy and a nonlinear potential term,

$$\hat{\mathcal{H}} = \frac{1}{2} {}^{\rm t}\hat{\boldsymbol{p}} \boldsymbol{A}^2 \hat{\boldsymbol{p}} + V(\hat{\boldsymbol{q}}). \tag{28}$$

The real matrix $\boldsymbol{A}^2 = \{A_{ij}^2\}$ is symmetric, ${}^{\rm t}\boldsymbol{A} = \boldsymbol{A}$, and positive definite. The influence action for anomalous dissipation takes the form

$$S_{\rm I}[\boldsymbol{p}(u)] = \int_0^{\beta} \frac{du}{2} \int_0^{\beta} \frac{du'}{\beta} {}^{\rm t}\boldsymbol{p}(u) \boldsymbol{K}(u-u') \boldsymbol{p}(u'), \tag{29}$$

where the kernel matrix $\boldsymbol{K}(u) = \{\kappa_{ij}(u)\}$ is a real symmetric matrix that replaces the scalar kernel $\kappa(u)$ of the single-particle case; as a function of u it keeps its symmetry and periodicity, $\boldsymbol{K}(u) = \boldsymbol{K}(-u) = \boldsymbol{K}(\beta - u)$, and satisfies $\int_0^{\beta} du \boldsymbol{K}(u) = 0$. For instance, in the case of N independent identical baths coupled to each momentum $\hat{p}_i$ one simply has a diagonal kernel, $\kappa_{ij}(u) = \delta_{ij}\kappa(u)$; this case will be considered for the application shown in Sec. IV C.

The Weyl symbol for system's density matrix is expressed as an N-dimensional integral,

$$\rho(\boldsymbol{p}, \boldsymbol{q}) = \int d\overline{\boldsymbol{q}}\, \overline{\rho}(\boldsymbol{p}, \boldsymbol{q}; \overline{\boldsymbol{q}}), \tag{30}$$

in terms of the reduced density collecting the contributions of paths sharing the average configuration $\overline{\boldsymbol{q}}$,

$$\overline{\rho}(\boldsymbol{p},\boldsymbol{q};\overline{\boldsymbol{q}})=\int \mathcal{D}[\boldsymbol{p}_u,\boldsymbol{q}_u]\delta\left(\overline{\boldsymbol{q}}-\int_0^\beta \frac{du}{\beta}\boldsymbol{q}_u\right)$$
$$\times\exp\{-\mathcal{S}[\boldsymbol{p}_u,\boldsymbol{q}_u;\boldsymbol{p},\boldsymbol{q}]-S[\boldsymbol{p}_u,\boldsymbol{q}_u;\overline{\boldsymbol{q}}]-S_{\mathrm{I}}[\boldsymbol{p}_u]\};\tag{31}$$

here the action is separated in a part containing the external variables $(\boldsymbol{p},\boldsymbol{q})$,

$$\mathcal{S}[\boldsymbol{p}_u,\boldsymbol{q}_u;\boldsymbol{p},\boldsymbol{q}]=\int_0^\beta du\left[\frac{i}{2}({}^{\mathrm{t}}\boldsymbol{q}_u\dot{\boldsymbol{p}}_u-{}^{\mathrm{t}}\boldsymbol{p}_u\dot{\boldsymbol{q}}_u)\right]$$
$$+\frac{i}{2}({}^{\mathrm{t}}\boldsymbol{q}_0\boldsymbol{p}_\beta-{}^{\mathrm{t}}\boldsymbol{p}_0\boldsymbol{q}_\beta)$$
$$+i[{}^{\mathrm{t}}(\boldsymbol{q}_\beta-\boldsymbol{q}_0)\boldsymbol{p}-{}^{\mathrm{t}}(\boldsymbol{p}_\beta-\boldsymbol{p}_0)\boldsymbol{q}],\tag{32}$$

and the part containing the system's Hamiltonian,

$$S[\boldsymbol{p}_u,\boldsymbol{q}_u;\overline{\boldsymbol{q}}]=\int_0^\beta du\left[\frac{1}{2}{}^{\mathrm{t}}\boldsymbol{p}_u\boldsymbol{A}^2\boldsymbol{p}_u+V(\boldsymbol{q}_u)\right].\tag{33}$$

In order to evaluate the effective potential, the trial action S_0 is defined as S by replacing $V(\boldsymbol{q}_u)$ by

$$V_0(\boldsymbol{q}_u;\overline{\boldsymbol{q}})=w(\overline{\boldsymbol{q}})+\frac{1}{2}{}^{\mathrm{t}}(\boldsymbol{q}_u-\overline{\boldsymbol{q}})\boldsymbol{B}^2(\overline{\boldsymbol{q}})(\boldsymbol{q}_u-\overline{\boldsymbol{q}}),\tag{34}$$

with a real symmetric matrix $\boldsymbol{B}^2(\overline{\boldsymbol{q}})$. The calculation of the corresponding $\overline{\rho}_0$, reported in Appendix E, is not a trivial extension of that for one degree of freedom, but can take advantage of the results obtained in Ref. [7] for the standard SPR model.

The simplest way for writing the final result is to give the expression of the thermal average of a generic observable $\hat{\mathcal{O}}$ in terms of its Weyl symbol $\mathcal{O}(\boldsymbol{p},\boldsymbol{q})$. This fundamental formula approximates quantum averages by means of a classical-like expression with the effective potential V_{eff},

$$\langle\hat{\mathcal{O}}\rangle=\frac{1}{\mathcal{Z}}\left(\frac{1}{2\pi\beta}\right)^{N/2}\frac{1}{\det\boldsymbol{A}}\int d\overline{\boldsymbol{q}}e^{-\beta V_{\mathrm{eff}}(\overline{\boldsymbol{q}})}\langle\!\langle\mathcal{O}(\boldsymbol{p},\overline{\boldsymbol{q}}+\boldsymbol{\xi})\rangle\!\rangle,\tag{35}$$

where $\langle\!\langle\cdot\rangle\!\rangle$ is the Gaussian average over the variables $\boldsymbol{p}$ and $\boldsymbol{\xi}$ determined by $\overline{\rho}_0$, as reported in Eq. (E10), and can be uniquely defined through its moments

$$\langle\!\langle\boldsymbol{\xi}\,{}^{\mathrm{t}}\boldsymbol{\xi}\rangle\!\rangle=\boldsymbol{C}(\overline{\boldsymbol{q}})=\frac{2}{\beta}\sum_{n=1}^{\infty}\boldsymbol{B}^{-1}\frac{\boldsymbol{\Psi}_n}{\nu_n^2+\boldsymbol{\Psi}_n}\boldsymbol{B}^{-1},\tag{36}$$

$$\langle\!\langle\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{p}\rangle\!\rangle=\boldsymbol{\Lambda}(\overline{\boldsymbol{q}})=\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\boldsymbol{B}\frac{1}{\nu_n^2+\boldsymbol{\Psi}_n}\boldsymbol{B},\tag{37}$$

whose components are the *renormalization coefficients* and

$$\boldsymbol{\Psi}_n(\overline{\boldsymbol{q}})=\boldsymbol{B}(\boldsymbol{A}^2+\boldsymbol{K}_n)\boldsymbol{B}.\tag{38}$$

The effective potential reads

$$V_{\mathrm{eff}}(\overline{\boldsymbol{q}})=w(\overline{\boldsymbol{q}})+\frac{1}{\beta}\mu(\overline{\boldsymbol{q}}),\tag{39}$$

with

$$\mu(\overline{\boldsymbol{q}})=\sum_{n=1}^{\infty}\ln\frac{\det(\nu_n^2+\boldsymbol{\Psi}_n)}{\nu_n^{2N}}.\tag{40}$$

To determine the parameters $w(\overline{q})$ and $\boldsymbol{B}(\overline{\boldsymbol{q}})$ we require, as in Eqs. (23) and (24), that the parameters of the trial action are such to match the $\overline{\rho}_0$ averages of the original and the trial potential, and the same for their second derivatives:

$$\langle\!\langle V(\overline{\boldsymbol{q}}+\boldsymbol{\xi})\rangle\!\rangle=w(\overline{\boldsymbol{q}})+\frac{1}{2}\mathrm{Tr}[\boldsymbol{B}^2(\overline{\boldsymbol{q}})\boldsymbol{C}(\overline{\boldsymbol{q}})],\tag{41}$$

$$\langle\!\langle\partial_{q_i}\partial_{q_j}V(\overline{\boldsymbol{q}}+\boldsymbol{\xi})\rangle\!\rangle=B_{ij}^2(\overline{\boldsymbol{q}}).\tag{42}$$

The latter equation together with Eq. (36) self-consistently determines the solution for the matrices $\boldsymbol{B}(\overline{\boldsymbol{q}})$ and $\boldsymbol{C}(\overline{\boldsymbol{q}})$. Its matrix character makes it useful to introduce the "low-coupling" approximation (LCA), in the very same way of Ref. [7], so it is sufficient to write here the final results. If $\overline{\boldsymbol{q}}_0$ is the configuration that minimizes $V_{\mathrm{eff}}(\overline{\boldsymbol{q}})$, one has the LCA effective potential

$$V_{\mathrm{eff}}(\overline{\boldsymbol{q}})=e^{\Delta}V(\overline{\boldsymbol{q}})-\Delta e^{\Delta}V(\overline{\boldsymbol{q}}_0)+\beta^{-1}\mu,\tag{43}$$

where $\mu=\mu(\overline{\boldsymbol{q}}_0)$ and the operator

$$\Delta=\frac{1}{2}\sum_{ij}C_{ij}\partial_{\overline{q}_i}\partial_{\overline{q}_j},\tag{44}$$

with $\boldsymbol{C}\equiv\boldsymbol{C}(\overline{\boldsymbol{q}}_0)$, is such that $e^{\Delta}V(\overline{\boldsymbol{q}})=\langle\!\langle V(\overline{\boldsymbol{q}}+\boldsymbol{\xi})\rangle\!\rangle$ (within the LCA). Therefore, Eqs. (36) and (42) have to be solved only for the minimum configuration, with a great simplification. In the simplest case of translation invariance an orthogonal Fourier transformation $\boldsymbol{U}=\{U_{ki}\}$ diagonalizes all matrices,

$$m_k^{-1}\delta_{kk'}=\sum_{ij}U_{ki}U_{k'j}A_{ij}^2,\tag{45}$$

$$m_k\omega_k^2\delta_{kk'}=\sum_{ij}U_{ki}U_{k'j}B_{ij}^2,\tag{46}$$

$$\kappa_{n,k}\delta_{kk'}=\sum_{ij}U_{ki}U_{k'j}\kappa_{n,ij},\tag{47}$$

so that

$$\mu=\sum_k\sum_{n=1}^{\infty}\ln\frac{\nu_n^2+(1+m_k\kappa_{n,k})\omega_k^2}{\nu_n^2};\tag{48}$$

the renormalization coefficients of Eqs. (36) and (37) become

$$\Lambda_{ij}=\sum_k U_{ki}U_{kj}\lambda_k, \quad C_{ij}=\sum_k U_{ki}U_{kj}\alpha_k, \tag{49}$$

where the variances of the kth mode,

$$\lambda_k=\frac{m_k}{\beta}\sum_{n=-\infty}^{\infty}\frac{\omega_k^2}{\nu_n^2+(1+m_k\kappa_{n,k})\omega_k^2}, \tag{50}$$

$$\alpha_k=\frac{2}{\beta m_k}\sum_{n=1}^{\infty}\frac{1+m_k\kappa_{n,k}}{\nu_n^2+(1+m_k\kappa_{n,k})\omega_k^2}, \tag{51}$$

generalize those of Eqs. (19) and (20); for instance, the ''on-site'' renormalization coefficient $D\equiv C_{ii}$ can be simply expressed as $D=\langle\langle \xi_i^2\rangle\rangle=N^{-1}\Sigma_k\alpha_k$. The partition function and thermal averages are to be evaluated by means of Eq. (35), where, of course, the effective potential and the double-bracket average are to be understood as the LCA ones. Note that, since the LCA Λ's do not depend on $\overline{\boldsymbol{q}}$, averages of observables involving only momenta are trivially evaluated as $\langle\mathcal{O}(\hat{\boldsymbol{p}})\rangle=\langle\langle\mathcal{O}(\boldsymbol{p})\rangle\rangle$.

IV. APPLICATIONS

A. Drude response model

In the case of *Ohmic dissipation* the memory is Markovian, $\eta(t)=\eta\delta(t-0)$, corresponding to $\tilde{\eta}(z)=\eta=$const and then to $\kappa_n\propto|n|$, so one can see that the coordinate fluctuation $\alpha(\overline{q})$, Eq. (20), is divergent, as well as $\mu(\overline{q})$, Eq. (18). This is due to the unphysical assumption of a vanishing response time from the dissipation bath; in the standard SPR model such divergence [3,5–7] only affects the momentum fluctuation, leaving the possibility to meaningfully evaluate thermal averages of coordinate-dependent quantities [6]. As when considering momentum-dependent quantities [7] in the standard case, in the anomalous SPR model we are therefore forced to take into account the finite bath's response time.

The Drude model of an exponentially decaying memory,

$$\eta(t)=\Theta(t)\,\eta\omega_{\mathrm{D}}e^{-\omega_{\mathrm{D}}t}, \quad \tilde{\eta}(z)=\frac{\eta\omega_{\mathrm{D}}}{\omega_{\mathrm{D}}+z} \tag{52}$$

is the simplest one that is physically meaningful: the constant η characterizes the strength of the coupling with the dissipation bath, while the Drude frequency ω_{D} characterizes its bandwidth (ω_{D}^{-1} is the response time); the Ohmic limit occurs for $\omega_{\mathrm{D}}\to\infty$.

B. Single particle in the double well

Given a potential $V(q)$, it is convenient to provide a dimensionless formulation by identifying characteristic energy and length scales, ϵ and σ. The first one could be a barrier height or a well depth, the latter is such that the change of V over its scale is comparable to ϵ. The dimensionless coordinate is $\hat{x}=\hat{q}/\sigma$ and the dimensionless potential $v(x)$ is such that $V(\sigma x)=\epsilon v(x)$. If x_m is the absolute minimum of $v(x)$ and $v''\equiv v''(x_m)$, the harmonic approximation (HA) of the system is characterized by the frequency $\omega_0=\sqrt{\epsilon v''/m\sigma^2}$; the ratio between the HA quantum energy level splitting $\hbar\omega_0$ and the overall energy scale ϵ defines the dimensionless coupling parameter

$$g=\frac{\hbar\omega_0}{\epsilon}=\sqrt{\frac{\hbar^2 v''}{m\epsilon\sigma^2}}. \tag{53}$$

Therefore, the system is weakly (strongly) ''quantum'' when g is small (large) compared to 1. Defining the dimensionless momentum $\hat{p}_x=\sigma\hat{p}/\hbar$, such that $[\hat{x},\hat{p}_x]=i$, the nondissipative part of the Hamiltonian (4) reads

$$\frac{\hat{\mathcal{H}}_{\mathrm{S}}}{\epsilon}=\frac{g^2}{v''}\frac{\hat{p}_x^2}{2}+v(\hat{x}). \tag{54}$$

In what follows, energies are given in units of ϵ, lengths in units of σ, frequencies in units of ω_0, and so on; the dimensionless temperature is $t=1/(\epsilon\beta)$.

As for the dissipation part, to make contact with the phenomenological behavior, note that the asymptotic damping function from the classical limit of the pseudo-Langevin equation (5) is $V''\tilde{\eta}(z)=m\omega_0^2\tilde{\eta}(z)$ and has the dimension of a frequency [it is indeed the counterpart of $\tilde{\gamma}(z)/m$ of the standard SPR model], so for the Drude form (52) we will deal with the corresponding dimensionless input parameters

$$\hat{\eta}=m\omega_0\eta, \quad \hat{\omega}_{\mathrm{D}}=\omega_{\mathrm{D}}/\omega_0. \tag{55}$$

The dimensionless term $m\kappa_n$ reads then

$$m\kappa_n=\hat{\eta}\hat{\omega}_{\mathrm{D}}\frac{\pi n}{\pi n+f_{\mathrm{D}}}\equiv a_n, \tag{56}$$

where $f_{\mathrm{D}}=\beta\hbar\omega_{\mathrm{D}}/2=(g/2t)\hat{\omega}_{\mathrm{D}}$.

Using the dimensionless parameter

$$f(x)=\frac{\beta\hbar\omega(x)}{2}=\frac{g}{2t}\frac{\omega(x)}{\omega_0} \tag{57}$$

in the place of $\omega(x)$, from the definitions (18), (19), and (20) we obtain

$$\mu(x)=\sum_{n=1}^{\infty}\ln\frac{(\pi n)^2+(1+a_n)f^2(x)}{(\pi n)^2}, \tag{58}$$

$$\lambda(x)=\frac{v''t}{g^2}\sum_{n=-\infty}^{\infty}\frac{f^2(x)}{(\pi n)^2+(1+a_n)f^2(x)}, \tag{59}$$

$$\alpha(x)=\frac{g^2}{2v''t}\sum_{n=1}^{\infty}\frac{1+a_n}{(\pi n)^2+(1+a_n)f^2(x)}, \tag{60}$$

where $\lambda=\langle\langle p_x^2\rangle\rangle$.

The example we choose here is the double-well potential,

$$v(x)=(1-x^2)^2, \tag{61}$$

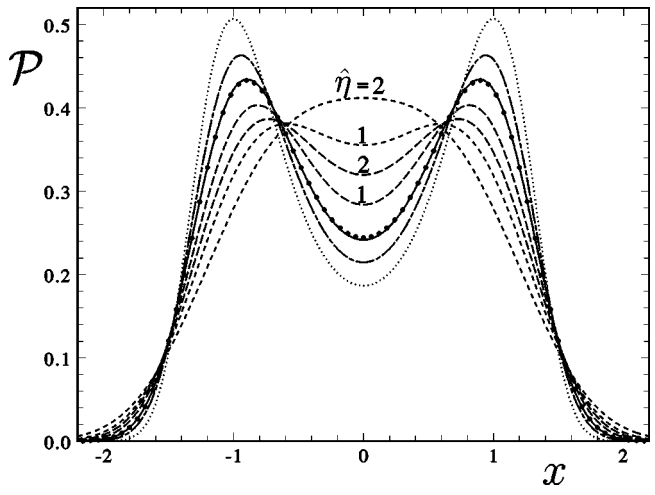

FIG. 1. Configuration density $\mathcal{P}(x)=\langle\delta(\hat{x}-x)\rangle$ of the double-well quartic potential for $g=2$, $t=1$, and different values of the damping parameters: $\hat{\eta}=1,2$ (as indicated on the picture), and $\hat{\omega}_{\mathrm{D}}=1$ (long-dashed lines) and 5 (short-dashed lines). The solid line is the nondissipative effective potential result for $\hat{\eta}=0$, that agrees with the corresponding exact result (filled circles); the dotted curve corresponds to the classical limit, and the dash-dotted curve reports the result of the standard SPR model from Ref. [6] for an Ohmic dissipative strength $\hat{\gamma}\equiv\gamma/(m\omega_0)=5$.

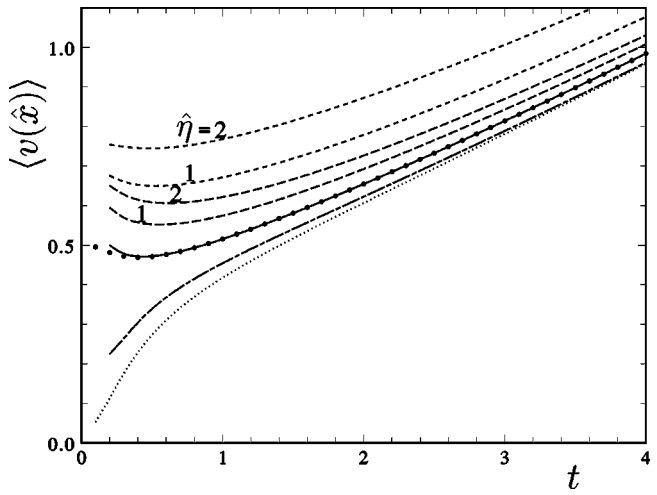

FIG. 2. Average potential energy $\langle\delta(\hat{x})\rangle$ of the double-well quartic potential vs reduced temperature t, for $g=2$ and different values of $\hat{\eta}$ and $\hat{\omega}_{\mathrm{D}}$. Curves and symbols as in Fig. 1.

which has two degenerate symmetric minima in $x_{\mathrm{m}}=\pm1$ with $v''=8$. From Eqs. (27) and (24) we obtain

$$v_{\mathrm{eff}}(x)=(1-x^2)^2-3\alpha^2(x)+t\mu(x), \tag{62}$$

$$f^2(x)=\frac{g^2}{8t^2}[3x^2+3\alpha(x)-1]. \tag{63}$$

Equation (63) has to be solved self-consistently with Eq. (60): we did this numerically; exact reference data can be obtained only for the nondissipative limit $\hat{\eta}=0$, e.g., by numerically solving the Schrödinger equation.

In Fig. 1 we report the shape of the coordinate probability distribution $\mathcal{P}(x)=\langle\delta(\hat{x}-x)\rangle$, for selected values of the dissipation strength $\hat{\eta}$ and bandwidth $\hat{\omega}_{\mathrm{D}}$, at the coupling $g=2$; this gives a ground state energy $e_0=0.697\,97$, with the next excited level at $e_1=1.046\,37$, above the barrier. When dissipation is switched on, $\mathcal{P}(x)$ tends to go farther away from the classical distribution $\mathcal{P}_{\mathrm{c}}\sim e^{-v(x)/t}$ due indeed to the enhanced fluctuations of the coordinate, and the two-peaked structure is eventually lost. This happens by rising either $\hat{\eta}$ or $\hat{\omega}_{\mathrm{D}}$: note the strong influence of the latter in increasing the fluctuations, which diverge in the Ohmic limit $\hat{\omega}_{\mathrm{D}}\to\infty$. The figure also reports the opposite result for the standard dissipative model, obtained as in Ref. [6], which clearly shows the suppression of the coordinate fluctuations towards the classical distribution.

Typical results found for the average potential energy $v(t)=\langle v(x)\rangle$ are displayed in Fig. 2. In the nondissipative case, comparison with the exact data shows that the effective potential gives very accurate results, in spite of the strong coupling. The curves for the anomalous dissipative system show an increase of the potential energy, corresponding to the higher fluctuations of the coordinate, while for the standard SPR model the opposite occurs.

C. ϕ^4 chain

The quantum ϕ^4 chain with standard dissipation was studied in Ref. [7]; the effective potential approach was previously applied in the nondissipative case [17,12] and checked through quantum Monte Carlo simulations [18]. It can be viewed as the discretized version of a continuum nonlinear field theory, described by the (undamped) Lagrangian

$$\mathcal{L}=Ga\sum_{i=1}^{N}\left[\frac{\dot{q}_i^2}{2}-\frac{c^2}{2a^2}(q_i-q_{i-1})^2+\frac{\Omega^2}{8}v(q_i)\right], \tag{64}$$

where a is the chain spacing, c is the ''relativistic'' velocity, Ω is the gap of the bare dispersion relation, G is a constant with the dimension (mass$\times$length), and periodic boundary conditions ensure translation symmetry. The local nonlinear potential

$$v(x)=(1-x^2)^2 \tag{65}$$

has two wells in $x_{\mathrm{m}}=\pm1$ [with $v''(x_{\mathrm{m}})=8$], so that the classical ϕ^4 chain has two degenerate translation-invariant minimum configurations, $\{q_i=1\}$ and $\{q_i=-1\}$, as well as relative minima connecting the two wells, the static ''kinks.'' In the continuum limit $ia\to z$ the kink configuration is indeed $q(z)=\pm\tanh[\Omega(z-z_0)/c]$, so that it is localized with a characteristic length c/Ω and its energy is $\varepsilon_{\mathrm{K}}=2G\Omega c/3$.

The ratio between the characteristic frequency Ω of the quasiharmonic excitations of the system and the energy scale ε_{K} defines the quantum coupling parameter $Q=\Omega/\varepsilon_{\mathrm{K}}=3/(2Gc)$; the discreteness of the chain is measured by the kink length in lattice units, $R=c/(\Omega a)$ ($R\to\infty$ in the continuum limit). Moreover, $t\equiv T/\varepsilon_{\mathrm{K}}$ is the reduced temperature used from now on. The most interesting features appear

when kinks are excited in the system; for instance, they cause a peak in the classical specific heat at $t \sim 0.2$.

Using the dimensionless quantities just introduced, the Weyl symbol for the undamped ϕ^4 Hamiltonian can be written as

$$\frac{\mathcal{H}}{\varepsilon_{\mathrm{K}}} = \frac{Q^2 R}{3} \sum_{i=1}^{N} p_i^2 + V(\boldsymbol{q}) \tag{66}$$

$$\frac{V(\boldsymbol{q})}{\varepsilon_{\mathrm{K}}} = \frac{3}{2R} \sum_{i=1}^{N} \left[\frac{R^2}{2} (q_i - q_{i-1})^2 + \frac{v(q_i)}{8} \right], \tag{67}$$

where the momenta are such that $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$, and Eq. (45) gives $m_k^{-1} = 2Q^2 R \varepsilon_{\mathrm{K}}/3 \equiv m^{-1}$.

As for dissipation, N identical independent environmental baths coupled to each momentum are assumed, giving a diagonal kernel matrix $\kappa_{n,ij} = \delta_{ij}\kappa_n$, so Eq. (47) gives $\kappa_{n,k} = \kappa_n$. We take $\kappa_n = |\nu_n| \tilde{\eta}(|\nu_n|)$ with the Drude form (52) for $\tilde{\eta}$; the dimensionless quantity $m_k \kappa_{n,k} = m\kappa_n \equiv a_n$ can eventually be written as in Eq. (56), so that the dissipation is characterized by the dimensionless parameters $\hat{\omega}_{\mathrm{D}} = \omega_{\mathrm{D}}/\Omega$ and $\hat{\eta} = 3\eta/(2QR)$

From Eqs. (42) and (46) the LCA renormalized frequencies are found to be

$$\omega_k^2 = \Omega^2 [1 - 3D(t) + 4R^2 \sin^2(k/2)], \tag{68}$$

while the relevant renormalization coefficient $D(t) \equiv C_{ii}$ follows from Eqs. (49) and (51),

$$D(t) = \frac{Q^2 R}{3t} \frac{1}{N} \sum_k \sum_{n=1}^{\infty} \frac{1 + a_n}{(\pi n)^2 + (1 + a_n) f_k^2}, \tag{69}$$

where $f_k = \beta\omega_k/2 = (Q/2t)[1 - 3D(t) + 4R^2 \sin^2(k/2)]$; these two self-consistent equations can be solved numerically, and it appears that the approximation is meaningful when $3D \ll 1$. Eventually, the LCA effective potential can be written as the original one, just replacing the local interaction $v(x)$ by

$$v_{\mathrm{eff}}(x) = [1 - 3D(t) - x^2]^2 + 6D^2(t) + t\tilde{\mu}(t), \tag{70}$$

with

$$\tilde{\mu}(t) = \frac{16R}{3} \frac{1}{N} \sum_k \sum_{n=1}^{\infty} \ln \frac{(\pi n)^2 + (1 + a_n) f_k^2}{(\pi n)^2}. \tag{71}$$

For any parameter set (t, Q, R, $\hat{\eta}$, $\hat{\omega}_{\mathrm{D}}$) the self-consistent computation of D and hence of the last term of $\tilde{\mu}$, which takes a negligible computer time using a continuum termination of the n summation, completely determines the effective potential $v_{\mathrm{eff}}(x)$. The problem is then ready for a numerical evaluation of Eq. (35), which gives the partition function (setting $\mathcal{O} = 1$) and the thermal averages of observables. We employed the numerical transfer matrix technique [19] that reduces the configuration integral for a one-dimensional array with nearest-neighbor interaction to a secular integral equation. The evaluation of the latter is implemented numerically using a discrete mesh for the values of each degree of freedom. Temperature scans over the region of interest were performed and several thermodynamic quantities were calculated, taking the value "per site" for the extensive ones. In particular, the internal energy $u(t) = f(t) - t\partial_t f(t)$ and the specific heat $c(t) = \partial_t u(t) = -t\partial_t^2 f(t)$ follow from the free energy per site, $f(t) = -N^{-1} t \ln \mathcal{Z}(t)$ by numerical derivation, while Eq. (35) was used to calculate the thermal average of the squared site coordinate $\langle \hat{q}_i^2 \rangle$ and of the local potential $\langle v(\hat{q}_i) \rangle$, the square nearest-neighbor displacement $\langle (\hat{q}_i - \hat{q}_{i-1})^2 \rangle$, and the square momentum $\langle \hat{p}_i^2 \rangle$. The quantities reported in Figs. 3, 4, and 5 are evaluated for fixed values of the kink length $R = 5$ and of the quantum coupling $Q = 0.2$, which give fairly strong quantum effects. For comparison, we also report the classical result corresponding to $Q = 0$. Moreover, a fixed representative value of the Drude cutoff frequency $\hat{\omega}_{\mathrm{D}} = 1$ is used, in order to analyze the dependence upon the dissipation strength $\hat{\eta}$.

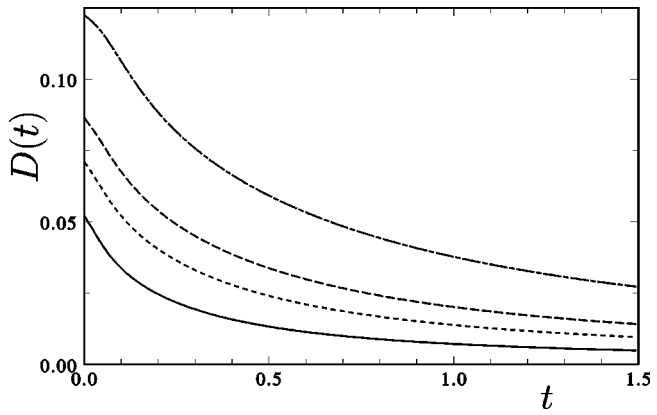

FIG. 3. Renormalization coefficient $D(t)$ [Eq. (69)] vs reduced temperature t, for different values of the damping strength $\hat{\eta}$, at fixed coupling parameter $Q = 0.2$, kink length $R = 5$, and Drude cutoff frequency $\hat{\omega}_{\mathrm{D}} = 1$. Solid line, $\hat{\eta} = 0$ (undamped); short-dashed line, $\hat{\eta} = 1$; long-dashed line, $\hat{\eta} = 2$; dash-dotted line, $\hat{\eta} = 5$. Note that $D(t)$ increases when $\hat{\eta}$ increases.

As we already remarked, the pure-quantum fluctuations of the coordinate are made stronger by the anomalous dissipative coupling. This appears in Fig. 3, where the pure-quantum renormalization coefficient $D(t) = \langle\langle \xi_i^2 \rangle\rangle$ of Eq. (69) is reported for different values of $\hat{\eta}$, starting from the nondissipative value $\hat{\eta} = 0$. The temperature dependence of the coefficient $D(t)$ affects the effective potential and the free energy, allowing us to describe how the classical behavior of thermal averages is affected by both quantum and dissipative effects.

In Fig. 4 the temperature behavior of the mean-square fluctuations of the site coordinate $\langle \hat{q}_i^2 \rangle$ is reported. At $t = 0$, in the classical case the coordinate lies in the minima $\{q_i^2 = 1\}$ of the potential (67), while in the quantum nondissipative case the value at $t = 0$ is smaller, $\langle \hat{q}_i^2 \rangle = 1 - 3D \simeq 0.84$: this corresponds to the minima of $v_{\mathrm{eff}}(q)$ and reflects the fact that quantum fluctuations make the configuration to climb the barrier rather than the steeper walls. Switching on the temperature enhances the same effect and $\langle \hat{q}_i^2 \rangle$ decreases at

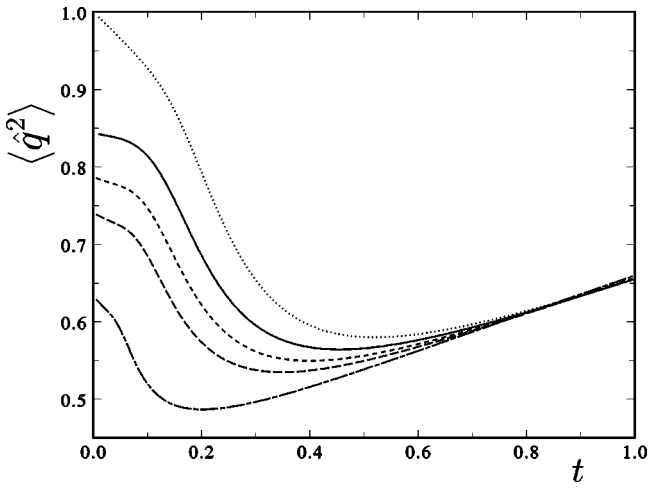

FIG. 4. Mean-square fluctuations of the coordinate operator $\langle \hat{q}_i^2 \rangle$ vs reduced temperature t. Parameters and lines as in Fig. 3. The dotted line represents the classical result ($Q=0$).

finite temperature, until when at $t \gtrsim 0.5$ many kinks are excited in the system and the coordinate distribution begins to spread towards the walls, causing the subsequent increase from a minimum value ~ 0.56 at $t \sim 0.45$, eventually collapsing onto the classical curve. At variance with the case of standard dissipation [7], when the damping strength $\hat{\eta}$ is switched on, a further enhancement of the distribution spread occurs: this could be qualitatively interpreted as an effective increase of the quantum coupling and allows $\langle \hat{q}_i^2 \rangle$ to reach smaller values, at smaller temperature.

On the other hand, the fluctuations of the momenta are quenched by dissipation, and the role of the damping effects is nonpredictable on a simple basis if one considers thermodynamic quantities where both coordinates and momenta enter into play. One of these is the specific heat $c(t)$. Its *nonlinear* part, namely, its total value minus the corresponding

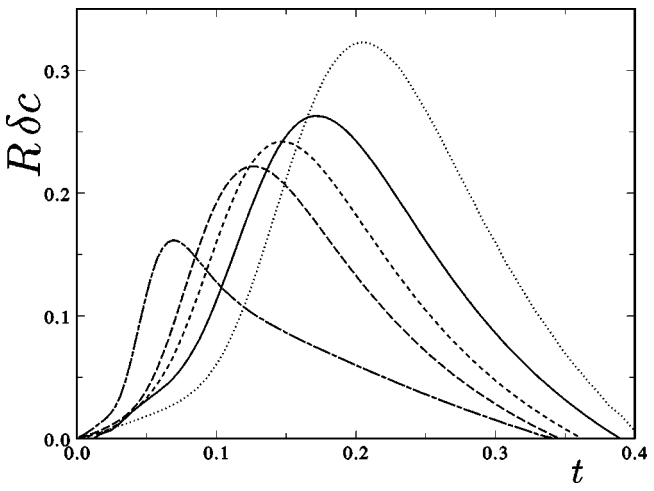

FIG. 5. Nonlinear contribution to the specific heat, defined as $\delta c(t) = c(t) - c_{\mathrm{H}}(t)$ (times the kink length R), vs reduced temperature t. Parameters and lines as in Fig. 3. The dotted line represents the classical result ($Q=0$). Increasing $\hat{\eta}$, $\delta c(t)$ goes farther away from the classical behavior, as if Q were raised [17].

(dissipative) harmonic contribution, $\delta c(t) = c(t) - c_{\mathrm{H}}(t)$, is very sensitive to the nonlinearity of the system, since its value is zero in a harmonic approximation. The fact that our approach retains all classical nonlinear features is crucial for getting δc, whose temperature behavior is reported in Fig. 5. Switching on and increasing the strength of the environmental coupling $\hat{\eta}$ the curves can be seen to go farther apart from the classical one, which is a nontrivial result. A physical explanation is that anomalous dissipation increases the uncertainty in the kink positions, substantially making them longer and effectively raising their mutual interaction.

V. CONCLUSIONS

In this paper we have discussed the problem of *anomalous dissipation*, previously considered by Leggett [9], and derived the effective potential formalism for an interacting many-body system with this kind of dissipation, which can arise from a microscopic description of the system-bath coupling, as done, for instance, in Ref. [10].

Our approach allows one to reduce quantum mechanical thermodynamic calculations to a classical-like configuration integral, where both quantum and dissipative effects are included. In order to deal with many degrees of freedom the necessary *low-coupling* approximation has been introduced. The latter, if the system's symmetries are exploited, results in very simple expressions for the renormalization coefficients appearing in the theory. This is shown in detail for the case of translation symmetry.

Applications of the framework have been made to the single particle in a double-well potential and to a discrete ϕ^4 one-dimensional field, whose strong nonlinearity yields kink excitations that play a dominant role. In both examples a Drude-like spectrum of the environmental coupling has been chosen for dissipation, and its influence on thermal quantities has been analyzed.

The effective potential is a unique tool for dealing with such a system, since any approximate theory must retain the strong nonlinearity, which mainly has a classical character, and this rules out conventional perturbative approaches. The method proved to be useful for the above model systems, so it is expected that it will find application in up-to-date physical problems. A possible application of the formalism regards the effect of the blackbody electromagnetic field [10] onto the phase diagram of two-dimensional Josephson junction arrays.

ACKNOWLEDGEMENT

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APPENDIX A: STANDARD SPR MODEL

Eliminating the bath variables from the equations of motion for the standard SPR model (1) one gets [4] the Langevin equation (2), where the coupling with the bath is described by the memory function $\gamma(t) = \Theta(t) \Sigma_\ell m_\ell \omega_\ell^2 \cos(\omega_\ell t)$, whose Laplace transform is

$$\tilde{\gamma}(z)=\sum_{\ell} m_{\ell}\omega_{\ell}^2\frac{z}{z^2+\omega_{\ell}^2}, \tag{A1}$$

and by the Gaussian random force $\hat{F}(t)$, whose time correlators can be expressed in terms of the memory function by spectral relations [4] depending on the bath's temperature.

The quantum thermodynamics, after the bath has been ''traced out'' in the path-integral for the system (1), is described by the density matrix [5]

$$\rho(q'',q')=\int_{q'}^{q''}\mathcal{D}[q(u)]\exp\{-S[q(u)]-S_{\mathrm{I}}(q(u))\}, \tag{A2}$$

$$S[q(u)]=\int_0^{\beta\hbar}\frac{du}{\hbar}\left[\frac{m}{2}\dot{q}^2(u)+V[q(u)]\right] \tag{A3}$$

and S_{I} given in Eq. (3); the Matsubara transform of the kernel $k(u)$ reads

$$k_n=\int_0^{\beta\hbar}\frac{du}{\beta\hbar}k(u)e^{-i\nu_n u}=\sum_{\ell} m_{\ell}\omega_{\ell}^2\frac{\nu_n^2}{\nu_n^2+\omega_{\ell}^2} \tag{A4}$$

and is thus given by $k_n=|\nu_n|\tilde{\gamma}(|\nu_n|)$. For a harmonic potential, $V(q)=m\omega^2q^2/2$, the evaluation of the path integral can be performed exactly and the density matrix turns out to be a Gaussian with partition function and variances as follows [5]:

$$\mathcal{Z}_{\mathrm{H}}=\frac{1}{\beta\hbar\omega}\prod_{n=1}^{\infty}\frac{\nu_n^2}{\nu_n^2+\omega^2+k_n/m},$$

$$\langle\hat{p}^2\rangle_{\mathrm{H}}=\frac{m}{\beta}\sum_{n=-\infty}^{\infty}\frac{\omega^2+k_n/m}{\nu_n^2+\omega^2+k_n/m},$$

$$\langle\hat{q}^2\rangle_{\mathrm{H}}=\frac{1}{\beta m}\sum_{n=-\infty}^{\infty}\frac{1}{\nu_n^2+\omega^2+k_n/m}. \tag{A5}$$

In the nondissipative limit, $k_n\to 0$, the summations give the well-known results $\mathcal{Z}_{\mathrm{H}}=1/(2\sinh f)$, $\langle\hat{q}^2\rangle_{\mathrm{H}}=(\hbar/2m\omega)\coth f$, and $\langle\hat{p}^2\rangle_{\mathrm{H}}=(\hbar m\omega/2)\coth f$, with $f=\beta\hbar\omega/2$.

APPENDIX B: ANOMALOUS SPR MODEL

From the Hamiltonian (4) we get the equations of motion

$$m\ddot{\hat{q}}+V'(\hat{q})=\sum_{\ell}\frac{m}{m_{\ell}}[m_{\ell}\omega_{\ell}^2\hat{q}_{\ell}-V'(\hat{q})], \tag{B1}$$

$$m_{\ell}(\ddot{\hat{q}}_{\ell}+\omega_{\ell}^2\hat{q}_{\ell})=V'(\hat{q}); \tag{B2}$$

the second one has the general solution $\hat{q}_{\ell}(t)=\hat{q}_{\ell}^{\mathrm{h}}(t)+\hat{q}_{\ell}^{\mathrm{p}}(t)$, with the retarded form of the particular solution

$$m_{\ell}\omega_{\ell}^2\hat{q}_{\ell}^{\mathrm{p}}(t)=V'(\hat{q}(t))-\int_{-\infty}^{t}dt'\cos[\omega_{\ell}(t-t')]\times\partial_{t'}V'(\hat{q}(t')), \tag{B3}$$

and the homogeneous solution

$$\hat{q}_{\ell}^{\mathrm{h}}(t)=\hat{q}_{\ell}\cos(\omega_{\ell}t)+\frac{\hat{p}_{\ell}}{m_{\ell}\omega_{\ell}}\sin(\omega_{\ell}t). \tag{B4}$$

Inserting this result in Eq. (B1) one gets Eq. (5), where

$$\eta(t)=\Theta(t)\sum_{\ell}\frac{1}{m_{\ell}}\cos(\omega_{\ell}t), \tag{B5}$$

while the fluctuating force term

$$\hat{F}(t)=m\sum_{\ell}\omega_{\ell}^2\hat{q}_{\ell}^{\mathrm{h}}(t) \tag{B6}$$

is a Gaussian random process, whose correlations arise from the assumption that the initial values $\{\hat{q}_{\ell},\hat{p}_{\ell}\}$ correspond to the equilibrium at the temperature $T=\beta^{-1}$ for the isolated bath, namely, $\langle\hat{q}_{\ell}\hat{q}_{\ell'}\rangle=\delta_{\ell\ell'}(\hbar/2m_{\ell}\omega_{\ell})\coth(\beta\hbar\omega_{\ell}/2)$, $\langle\hat{p}_{\ell}\hat{p}_{\ell'}\rangle=\delta_{\ell\ell'}(\hbar m_{\ell}\omega_{\ell}/2)\coth(\beta\hbar\omega_{\ell}/2)$, $\langle\hat{q}_{\ell}\hat{p}_{\ell'}\rangle=\delta_{\ell\ell'}i\hbar/2$.

The thermal density matrix for the harmonic oscillator, $V(q)=m\omega^2q^2/2$, can be evaluated reducing the problem to the standard-dissipation one, just by performing the canonical transformation $\hat{q}\to-\hat{p}/(m\omega)$ and $\hat{p}\to m\omega\hat{q}$; to get exactly the same form of Eq. (1) it is sufficient to replace $\hat{p}_{\ell}\to m\omega\hat{q}_{\ell}$, $\hat{q}_{\ell}\to-\hat{p}_{\ell}/(m\omega)$, and $m_{\ell}\to(m\omega)^2/(m_{\ell}\omega_{\ell}^2)$. One finds, of course, Eqs. (A5) with the exchange of the expressions $\langle\hat{q}^2\rangle_{\mathrm{H}}\to(m\omega)^{-2}\langle\hat{p}^2\rangle_{\mathrm{H}}$ and $\langle\hat{p}^2\rangle_{\mathrm{H}}\to(m\omega)^2\langle\hat{q}^2\rangle_{\mathrm{H}}$. Due to the above redefinition of the bath masses $\{m_{\ell}\}$, the kernel (A4) takes the form $k_n=(m\omega)^2\kappa_n$, with

$$\kappa_n=\sum_{\ell}\frac{1}{m_{\ell}}\frac{\nu_n^2}{\nu_n^2+\omega_{\ell}^2}, \tag{B7}$$

so eventually one has

$$\mathcal{Z}_{\mathrm{H}}=\frac{1}{\beta\hbar\omega}\prod_{n=1}^{\infty}\frac{\nu_n^2}{\nu_n^2+(1+m\kappa_n)\omega^2} \tag{B8}$$

$$\langle\hat{p}^2\rangle_{\mathrm{H}}=\frac{m}{\beta}\sum_{n=-\infty}^{\infty}\frac{\omega^2}{\nu_n^2+(1+m\kappa_n)\omega^2}, \tag{B9}$$

$$\langle\hat{q}^2\rangle_{\mathrm{H}}=\frac{1}{m\beta}\sum_{n=-\infty}^{\infty}\frac{1+m\kappa_n}{\nu_n^2+(1+m\kappa_n)\omega^2}. \tag{B10}$$

APPENDIX C: ANOMALOUS INFLUENCE ACTION

For the bath part of Eq. (4) the Feynman-Kac expression in terms of the momentum can be used: the definition of the influence action is, therefore,

$$e^{-S_{\rm I}[p(u)]}=\frac{1}{\mathcal{Z}_{\rm B}}\prod_{\ell}\ \oint \mathcal{D}[p_{\ell}(u)]\exp(-S_{\rm B}[\{p_{\ell}(u)\}]), \tag{C1}$$

where $\mathcal{Z}_{\rm B}=\Pi_{\ell}(2\sinh\omega_{\ell})^{-1}$ is the partition function of the isolated bath, and the bath action reads

$$S_{\rm B}=\int_0^{\beta\hbar}\frac{du}{\hbar}\sum_{\ell}\frac{1}{2m_{\ell}\omega_{\ell}^2}[\dot{p}_{\ell}^2(u)+\omega_{\ell}^2\{p_{\ell}(u)-p(u)\}^2]. \tag{C2}$$

Performing the Fourier transformation into Matsubara components $p(u)=\Sigma_n p_n e^{i\nu_n u}$ [and same for $p_{\ell}(u)$] one gets

$$\begin{aligned}S_{\rm B}&=\sum_{\ell}\frac{\beta}{2m_{\ell}\omega_{\ell}^2}\sum_n\ [\nu_n^2|p_{\ell n}|^2+\omega_{\ell}^2|p_{\ell n}-p_n|^2]\\&=\sum_{\ell}\frac{\beta}{2m_{\ell}\omega_{\ell}^2}\sum_n\left[(\nu_n^2+\omega_{\ell}^2)|\tilde{p}_{\ell n}|^2+\frac{\nu_n^2\omega_{\ell}^2}{\nu_n^2+\omega_{\ell}^2}|p_n|^2\right],\end{aligned} \tag{C3}$$

where the variable shift $\tilde{p}_{\ell n}=p_{\ell n}-[\omega_{\ell}^2/(\nu_n^2+\omega_{\ell}^2)]p_n$ has been performed. The path integral over $\tilde{p}_{\ell}(u)$ gives just the partition function $\mathcal{Z}_{\rm B}$, so one is left with the *anomalous* influence action

$$S_{\rm I}[p(u)]=\frac{\beta}{2}\sum_{n=-\infty}^{\infty}\kappa_n|p_n|^2, \tag{C4}$$

i.e., Eq. (6), where κ_n is again given by Eq. (B7), so that comparing with the Laplace transform of the memory function (B5),

$$\tilde{\eta}(z)=\sum_{\ell}\frac{1}{m_{\ell}}\frac{z}{z^2+\omega_{\ell}^2}, \tag{C5}$$

one has $\kappa_n=|\nu_n|\tilde{\eta}(|\nu_n|)$, i.e. Eq. (7).

APPENDIX D: EVALUATION OF $\bar{\rho}_0$ FOR ONE DEGREE OF FREEDOM

Let us start from the definition (14) of the reduced density in terms of the system's trial action, given by Eq. (9) with the trial potential (12),

$$\begin{aligned}\hbar S_0=&\int_0^{\beta\hbar}du\left[\frac{i}{2}(q_u\dot{p}_u-p_u\dot{q}_u)+\frac{p_u^2}{2m}+w+\frac{m\omega^2(q_u-\bar{q})^2}{2}\right]\\&+\frac{i}{2}(q_0p_\beta-p_0q_\beta)+i[(q_\beta-q_0)p-(p_\beta-p_0)q],\end{aligned} \tag{D1}$$

and of the dissipative action (6). In the calculations, $\bar{q}$ is to be regarded as a fixed parameter and we can omit to explicit the dependence of w and ω^2 on it. Performing the shift $q_u\to\bar{q}+q_u$ and using the Fourier representation

$$\delta\left(\int_0^{\beta\hbar}\frac{du}{\beta\hbar}q_u\right)=\beta m\omega^2\int\frac{dz}{2\pi}\exp\left(\int_0^{\beta\hbar}\frac{du}{\hbar}im\omega^2zq_u\right) \tag{D2}$$

one gets

$$\begin{aligned}\bar{\rho}_0(p,q;\bar{q})=&\beta m\omega^2e^{-\beta w}\int\frac{dz}{2\pi}\int\mathcal{D}[p_u,q_u]\\&\times\exp\{-S_1[p_u,q_u;\bar{q}]-S_{\rm I}[p_u]\},\end{aligned} \tag{D3}$$

$$\begin{aligned}\hbar S_1=&\int_0^{\beta\hbar}du\left[\frac{i}{2}(q_u\dot{p}_u-p_u\dot{q}_u)+\frac{p_u^2}{2m}+\frac{m\omega^2q_u^2}{2}-im\omega^2zq_u\right]\\&+\frac{i}{2}(q_0p_\beta-p_0q_\beta)+i[(q_\beta-q_0)p-(p_\beta-p_0)\xi],\end{aligned} \tag{D4}$$

where $\xi\equiv q-\bar{q}$. To eliminate the linear term we shift now $q_u\to q_u+iz$,

$$\begin{aligned}\bar{\rho}_0(p,q;\bar{q})=&\beta m\omega^2e^{-\beta w}\int\frac{dz}{2\pi}e^{-\beta m\omega^2z^2/2}\\&\times\int\mathcal{D}[p_u,q_u]\exp\{-S_2[p_u,q_u;\bar{q}]-S_{\rm I}[p_u]\},\end{aligned} \tag{D5}$$

so that S_2 is the harmonic-oscillator action,

$$\begin{aligned}\hbar S_2=&\int_0^{\beta\hbar}du\left[\frac{i}{2}(q_u\dot{p}_u-p_u\dot{q}_u)+\frac{p_u^2}{2m}+\frac{m\omega^2q_u^2}{2}\right]\\&+\frac{i}{2}(q_0p_\beta-p_0q_\beta)+i[(q_\beta-q_0)p\\&-(p_\beta-p_0)(\xi-iz)].\end{aligned} \tag{D6}$$

Now, the point is to get rid of the path integral appearing in Eq. (D5), which represents the Weyl symbol for the density operator of the harmonic oscillator with the dissipative action $S_{\rm I}[p_u]$. However, the result for the harmonic oscillator with anomalous dissipation is known from Eqs. (B8), (B9), and (B10),

$$\rho_{\rm H}(p,q)=\frac{2\pi e^{-\mu}}{\beta\omega}\frac{e^{-p^2/2\lambda_{\rm T}}}{\sqrt{2\pi\lambda_{\rm T}}}\frac{e^{-q^2/2\alpha_{\rm T}}}{\sqrt{2\pi\alpha_{\rm T}}}, \tag{D7}$$

where the partition function from Eq. (B8) is written as $\mathcal{Z}=e^{-\mu}/(\beta\hbar\omega)$ and μ given as in Eq. (18). One gets, therefore,

$$\overline{\rho_0}(p,q;\overline{q}) = \beta m\omega^2 e^{-\beta w} \int \frac{dz}{2\pi} e^{-\beta m\omega^2 z^2/2} \rho_{\mathrm{H}}(p,\xi - iz). \tag{D8}$$

This last Gaussian convolution affects the coordinate part and results in a Gaussian for ξ with the pure-quantum variance $\alpha = \alpha_{\mathrm{T}} - 1/(m\omega^2\beta)$ reported in Eq. (20): what is subtracted is just the classical contribution, i.e., the $n=0$ term in Eq. (B10). The final result is just Eq. (16).

APPENDIX E: EVALUATION OF $\overline{\rho_0}$ FOR MANY DEGREES OF FREEDOM

We evaluate here the path integral (31) with the trial action S_0 [i.e., Eq. (33) with the "potential" (34)]; the fixed argument of $w(\overline{q})$ and of the matrix $\boldsymbol{B}(\overline{q})$ is omitted in the following. First, the δ function is represented as

$$\delta\left(\overline{q} - \int_0^\beta \frac{du}{\beta} q_u\right) = \det(\beta \boldsymbol{B}^2) \int \frac{d\boldsymbol{z}}{(2\pi)^N} \times \exp\left\{ i \int_0^\beta du\, {}^{\mathrm{t}}\boldsymbol{z} \boldsymbol{B}^2 (q_u - \overline{q}) \right\}, \tag{E1}$$

and the exponential can be incorporated into the action S_0; then, performing the shift $q_u \to q_u + \overline{q} + i\boldsymbol{z}$ and introducing the fluctuation variable $\boldsymbol{\xi} = \boldsymbol{q} - \overline{\boldsymbol{q}}$ one gets

$$\overline{\rho_0}(\boldsymbol{p},\boldsymbol{q};\overline{\boldsymbol{q}}) = \frac{e^{-\beta w}}{\det \boldsymbol{C}_{\mathrm{C}}} \int \frac{d\boldsymbol{z}}{(2\pi)^N} \exp\left(-\frac{1}{2} {}^{\mathrm{t}}\boldsymbol{z} \boldsymbol{C}_{\mathrm{C}}^{-1} \boldsymbol{z} \right) \rho_{\mathrm{HA}}(\boldsymbol{p}, \boldsymbol{\xi} - i\boldsymbol{z}), \tag{E2}$$

where $\boldsymbol{C}_{\mathrm{C}}^{-1} \equiv \beta \boldsymbol{B}^2$ and

$$\rho_{\mathrm{HA}}(\boldsymbol{p},\boldsymbol{q}) = \int \mathcal{D}[\boldsymbol{p}_u, \boldsymbol{q}_u] \exp\{ -\mathcal{S}[\boldsymbol{p}_u, \boldsymbol{q}_u; \boldsymbol{p}, \boldsymbol{q}] - S_1[\boldsymbol{p}_u, \boldsymbol{q}_u; \overline{\boldsymbol{q}}] - S_{\mathrm{I}}[\boldsymbol{p}_u] \}, \tag{E3}$$

with

$$S_1 = \int_0^\beta \frac{du}{2} [\, {}^{\mathrm{t}}\boldsymbol{p}_u \boldsymbol{A}^2 \boldsymbol{p}_u + {}^{\mathrm{t}}\boldsymbol{q}_u \boldsymbol{B}^2 \boldsymbol{q}_u]. \tag{E4}$$

The path integral (E3) corresponds to the density matrix of a harmonic system with anomalous dissipation, $\rho_{\mathrm{HA}}(\boldsymbol{p},\boldsymbol{q})$. The canonical exchange $\boldsymbol{q}_u \to \boldsymbol{p}_u$ and $\boldsymbol{p}_u \to -\boldsymbol{q}_u$ has the effect of turning the dissipation into standard, $S_{\mathrm{I}}[\boldsymbol{p}_u] \to S_{\mathrm{I}}[\boldsymbol{q}_u]$; performing the same transformation onto the external variables $(\boldsymbol{p},\boldsymbol{q})$ and exchanging the matrices $\boldsymbol{A}^2 \leftrightarrow \boldsymbol{B}^2$, an exact correspondence with the standard dissipative harmonic system is established, $\rho_{\mathrm{HA}}(\boldsymbol{p},\boldsymbol{q}) = \rho_{\mathrm{HS}}(\boldsymbol{q}, -\boldsymbol{p})$, and the known result for a harmonic system with standard dissipation [7] can be used, eventually getting

$$\rho_{\mathrm{HA}}(\boldsymbol{p},\boldsymbol{q}) = \frac{e^{-\mu_{\mathrm{H}}}}{\det(\beta \boldsymbol{A}\boldsymbol{B})} \frac{\exp\left[-\frac{1}{2} {}^{\mathrm{t}}\boldsymbol{q} \boldsymbol{C}_{\mathrm{H}}^{-1} \boldsymbol{q} \right]}{\sqrt{(2\pi)^N \det \boldsymbol{C}_{\mathrm{H}}}} \frac{\exp\left[\frac{1}{2} {}^{\mathrm{t}}\boldsymbol{p} \boldsymbol{\Lambda}_{\mathrm{H}}^{-1} \boldsymbol{p} \right]}{\sqrt{(2\pi)^N \det \boldsymbol{\Lambda}_{\mathrm{H}}}}, \tag{E5}$$

where

$$\mu_{\mathrm{H}} = \sum_{n=1}^{\infty} \ln \frac{\det(\nu_n^2 + \boldsymbol{\Psi}_n)}{\nu_n^{2N}}, \tag{E6}$$

$$\boldsymbol{C}_{\mathrm{H}} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \boldsymbol{B}^{-1} \frac{\boldsymbol{\Psi}_n}{\nu_n^2 + \boldsymbol{\Psi}_n} \boldsymbol{B}^{-1}, \tag{E7}$$

$$\boldsymbol{\Lambda}_{\mathrm{H}} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \boldsymbol{B} \frac{1}{\nu_n^2 + \boldsymbol{\Psi}_n} \boldsymbol{B}, \tag{E8}$$

and

$$\boldsymbol{\Psi}_n = \boldsymbol{B}[\boldsymbol{A}^2 + \boldsymbol{K}_n]\boldsymbol{B}. \tag{E9}$$

Eventually, the Gaussian convolution in Eq. (E2) washes out the $n=0$ component of $\boldsymbol{C}$, leaving

$$\overline{\rho_0}(\boldsymbol{p},\boldsymbol{q};\overline{\boldsymbol{q}}) = \left(\frac{1}{2\pi\beta} \right)^{N/2} \frac{1}{\det \boldsymbol{A}} \exp\{ -\beta [w(\overline{\boldsymbol{q}}) + \mu(\overline{\boldsymbol{q}})] \} \times \frac{\exp\left[-\frac{1}{2} {}^{\mathrm{t}}\boldsymbol{q} \boldsymbol{C}^{-1} \boldsymbol{q} \right]}{\sqrt{(2\pi)^N \det \boldsymbol{C}}} \frac{\exp\left[\frac{1}{2} {}^{\mathrm{t}}\boldsymbol{p} \boldsymbol{\Lambda}^{-1} \boldsymbol{p} \right]}{\sqrt{(2\pi)^N \det \boldsymbol{\Lambda}}}, \tag{E10}$$

where $\boldsymbol{C} = \boldsymbol{C}_{\mathrm{H}} - \boldsymbol{C}_{\mathrm{C}}$ and $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\mathrm{H}}$.

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